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Blackhole evaporation—stress tensor approach

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Abstract The stress tenspr approach to black hole evaporation has been reviewed in this talk

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1. Introduction

As first pointed out by Hawking [1,2], the gravitational field of a collapsing object will induce the *quantum creation of particles* so that the object radiates with a thermal spectrum at a temperature inversely proportional to the mass of the object.

Earlier, calculations of this effect *examined the behaviour of the quantum fields only near infinity*. Consequently it was not clear precisely where the radiation is being created, and what is happening near the horizon of the "black hole". Davies, Fulling and Unruh [3] pointed out for the first time that a knowledge of the energy-momentum tensor of the quantum field in the vicinity of the object would help in clarifying the details of the creation process. Unfortunately, this quantity is *always formally divergent*, and the meaningful physical component must be extracted by a regularisation procedure. Such procedures always contain ambiguities which must be resolved by the application of additional criteria, such as physical reasonableness.

Besides the problems of regularisation, mathematical complexities have prevented detailed *discussion of quantum field theory near the surface of a blackhole*. However, it is possible to circumvent the latter problem by studying a simple two-dimensional model of the blackhole. *This model has the advantage of possessing a conformally flat metric so that* *the mode functions for the quantum field can be explicitly evaluated everywhere, while retaining the essential features of the Hawking evaporation process.* The highly plausible character of the "renormalised" energy-momentum tensor for this simple model encourages the hope that the qualitative features of the full four-dimensional collapse are contained in this treatment.

2. Stress tensor

The metric for any two-dimensional space-time is conformally flat and may be written as

$$ds^2 = C(u, v)\,du\,dv, \tag{1}$$

where u, v are null coordinates. The massless scalar field, ϕ, for this metric obeys the simple equation

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi = 0. \tag{2}$$

The solutions of this equation are

$$\phi = f(u) + g(v), \tag{3}$$

where $f(u)$ and $g(v)$ are, in general, arbitrary functions, restricted only by the spatial boundary conditions.

It is intended to calculate the expectation value of the operator

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{\alpha}_{,} \tag{4}$$

in some quantum state. In expanding the operator ϕ in normal modes, we assume that there exist null coordinates $\bar{u}$, $\bar{v}$ such that the ingoing and outgoing parts of a normal mode are respectively

$$e^{-i\omega\bar{v}}/(4\pi|\omega|)^{1/2}, \quad e^{-i\omega\bar{u}}/(4\pi|\omega|)^{1/2}. \tag{5}$$

The state which we have examined is the one annihilated by the operators with modes $\omega > 0$ in the field expansion.

If the geometry is initially static or has an asymptotically flat region at infinity, this state is made unique by the requirement that the modes reduce to ordinary plane waves in that region. This state is then that in which no particles are present initially (before the collapse begins as in the problem of Davies, Fulling and Unruh [3] discussed in § 3), and is conventionally called the "vacuum" or "in-vacuum" state,

On regularisation (on physical grounds), the expectation value of $T_{\mu\nu}$ in this state (also designated by $T_{\mu\nu}$) is

$$T_{\mu\nu} = \theta_{\mu\nu} + \frac{R}{48\pi}g_{\mu\nu} \tag{6}$$

where R is the curvature scalar and $\theta_{\mu\nu}$ as evaluated in the special $\bar{u}$, $\bar{v}$ coordinates has the components

$$\theta_{\bar{u}\bar{u}} = -(12\pi)^{-1} C^{\frac{1}{2}} (C^{-\frac{1}{2}})_{,\bar{u}\bar{u}},$$

$$\theta_{\bar{v}\bar{v}} = -(12\pi)^{-1} C^{\frac{1}{2}} (C^{-\frac{1}{2}})_{,\bar{v}\bar{v}}, \tag{7}$$

$$\theta_{\bar{u}\bar{v}} = \theta_{\bar{v}\bar{u}} = 0.$$

The regularisation scheme adopted for derivation of (6) stimulated some controversy, because it involves discarding certain ambiguous terms which inevitably arise as an artefact of the regularisation process (these terms are ambiguous because they depend on the direction of point-splitting). Because of this controversy, Davies [4] adopted an alternative procedure to confirm the result (6). Remarkably, however, it is possible to determine $<T_{\mu\nu}>$ uniquely without regularising infinite quantities at all, provided that we assume that the stress-tensor possesses a non-zero trace. Here, it is important to mention that in two dimensions, quite general arguments imply that conservation, zero trace and particle production are incompatible.

Let us consider the metric in $\bar{u}$, $\bar{v}$ coordinates in the conformally flat form

$$ds^2 = C(\bar{u}, \bar{v})\, d\bar{u}\, d\bar{v}. \tag{8}$$

The only non-vanishing Christoffel symbols are then

$$\Gamma^{\bar{u}}_{\bar{u}\bar{u}} = C^{-1}\partial_{\bar{u}} C, \quad \Gamma^{\bar{v}}_{\bar{v}\bar{v}} = C^{-1}\partial_{\bar{v}} C. \tag{9}$$

The stress-tensor $T_{\mu\nu}$ is defined to be covariantly conserved,

$$\nabla_{\mu} T^{\mu\nu} = 0, \tag{10}$$

which in terms of C becomes

$$\partial_{\bar{v}} T_{\bar{u}\bar{u}} + \frac{1}{4} C \partial_{\bar{u}} T^{\mu}{}_{\mu} = 0 \tag{11}$$

together with a similar expression for $T_{\bar{v}\bar{v}}$ with $\bar{u}$ and $\bar{v}$ interchanged.

The trace in (11) is assumed to be non-zero, even though the stress tensor *operator* for massless scalar fields is known to be traceless. The appearance of a trace in the vacuum expectation value of a (formally divergent) traceless operator is known as a conformal anomaly, because it breaks the conformal invariance. Conformal anomalies are to be expected on general grounds in quantum field theory [5]. Here we only need assume that $T^{\mu}{}_{\mu}$ is a non-vanishing local quantity. It is a scalar quantity with the dimensions (Length)$^{-2}$ (in units $\hbar = c = 1$) so it must consist of terms which are quadratic in derivatives of C. As there is no conformal anomaly in flat space-time, $T^{\mu}{}_{\mu}$ must vanish for certain choices of the conformal factor C. This requirement suffices to determine the trace to within an overall numerical factor. First, it is noted that, if C is a function of $\bar{u}$ or $\bar{v}$ alone, space-time is flat, because a simple rescaling of one null coordinate reduces the right-hand side of eq. (8) to $d\bar{u}\,d\bar{v}$. Hence $T^{\mu}{}_{\mu}$ can only contain a linear combination of the factors $\partial_{\bar{u}} C \partial_{\bar{v}} C$ and $\partial_{\bar{u}} \partial_{\bar{v}} C$. Because the theory does not contain a characteristic length, a simple scaling argument shows that $T^{\mu}{}_{\mu}$ must be a homogeneous functional in C of degree–1. Consequently

$$T^{\mu}_{\mu} = aC^{-2}\partial_{\bar{u}}\partial_{\bar{v}}C + b\partial_{\bar{u}}C\partial_{\bar{v}}C. \tag{12}$$

Next we note that the choice $C = e^{\bar{u}+\bar{v}}$ corresponds to the Milne Universe, which is just Minkowski space in disguise, so we require the right-hand side of (12) to vanish in this case. This fixes $a = -b$, so

$$T^{\mu}_{\mu} = -\frac{1}{4}aR \tag{13}$$

where R is the scalar curvature

Equation (11) may now be written in the form

$$\partial_{\bar{v}}T_{\bar{u}\bar{u}} = \frac{1}{2}a\partial_{\bar{v}}\left[C^{1/2}\partial^{2}_{\bar{u}}C^{-1/2}\right] \tag{14}$$

which may be immediately integrated to give

$$T_{\bar{u}\bar{u}} = \frac{1}{2}aC^{1/2}\partial^{2}_{\bar{u}}C^{-1/2} + f(\bar{u}), \tag{15}$$

where f is an arbitrary function of $\bar{u}$. To determine $f(\bar{u})$, it is noted first that, as $T_{\bar{u}\bar{u}}$ is local, f can depend on the geometry only through C and its derivatives at the point $(\bar{u},\bar{v})$ of interest. Now if $R \neq 0$, C will be a function of both $\bar{u}$ and $\bar{v}$, so $f(\bar{u})$ is generally independent of C because it is a function of $\bar{u}$ alone. At most f can be a constant. Davies [4] has omitted this constant.

To fix up the value of a, Davies has appealed to a special case, the case of a moving mirror, emitting radiation and obtained the value (with $f = 0$)

$$a = -(24\pi)^{-1} \tag{16}$$

Hence the complete stress tensor is

$$T_{\mu\nu} = \theta_{\mu\nu} + (48\pi)^{-1}Rg_{\mu\nu} \tag{17}$$

where $\theta_{\mu\nu}$ are as given in (7). $T_{\bar{v}\bar{v}}$ follows from $T_{\bar{u}\bar{u}}$ by interchange of $\bar{u}$ and $\bar{v}$ and the values of the conformal anomalies.

$$T^{\mu}_{\mu} = (24\pi)^{-1}R \tag{18}$$

No regularisation has been used to obtain these results.

3. Gravitational collapse of a spherical shell

Now (6) [or (17)] is applied to the collapse of a spherical shell. The two-dimensional metric is obtained by eliminating the angular coordinates from that of a four-dimensional shell of matter which collapses *at high velocity*. Inside the shell, the space-time is flat, whereas outside the shell the metric takes the Schwarzschild form

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2. \tag{19}$$

There exist three useful sets of null co-ordinates for this problem. In the first, given outside the shell by

$$\begin{aligned} u &= t - r^* \\ v &= t + r^* \\ r^* &= r + 2M\ln(r/2M - 1) \end{aligned} \tag{20}$$

the external metric takes the simple form

$$ds^2 = \left(1 - \frac{2M}{r}\right)du\,dv \tag{21}$$

where r is an implicit function of u, v by eqs. (20).

The second set, U, V, is defined so that the interior metric takes the simple form

$$dS^2 = dU dV. \tag{22}$$

The relation between the u, v and the U, V co-ordinates has been obtained by Unruh [6] by demanding continuity of the metric across the boundary.

Finally, we have the co-ordinates $\bar{u}$, $\bar{v}$ which are to appear in mode solutions (5) and in the determination [eq. (7)] of the energy-momentum tensor. Following Unruh [6], relations are obtained which lead to an expression for the external metric in $\bar{u}$, $\bar{v}$ co-ordinates and to values for $T_{\mu\nu}$. For retarded times $\bar{u}$ before the onset of the collapse, one obtains

$$ds^2 = \left(1 - \frac{2M}{r}\right)d\bar{u}\,d\bar{v} \tag{23}$$

that is, the conformal factor, $C(\bar{u}, \bar{v})$, to be used in eq. (7) is

$$C = 1 - \frac{2M}{r} \tag{24}$$

in the external region of space-time. The values of $T_{\mu\nu}$ in this region expressed in u, v coordinates are

$$T_{uu} = (24\pi)^{-1}\left(\frac{3M^2}{2r^4} - \frac{M}{r^3}\right).$$

$$T_{uv} = T_{vu} = (24\pi)^{-1}\left(\frac{2M^2}{r^4} - \frac{M}{r^3}\right), \tag{25}$$

$$T_{vv} = (24\pi)^{-1}\left(\frac{3M^2}{2r^4} - \frac{M}{r^3}\right).$$

For retarded times, u, long after the collapse, the external conformal factor in $\bar{u}, \bar{v}$ co-ordinates takes the form

$$C(\bar{u}, \bar{v}) = \left(1 - \frac{2M}{r}\right)\left[\frac{4M}{A-\bar{u}} + 0(1)\right], \tag{26}$$

where $0(1)$ are terms of order unity in $\bar{u}$ and A is a parameter such that $\bar{u} = A$ is the equation for the future horizon. Evaluating $T_{\mu\nu}$ outside the shell, transforming to u, v co-ordinates, and neglecting terms which die off for large values of u, one obtains

$$T_{uu} = (24\pi)^{-1}\left[\frac{3M^2}{2r^4} - \frac{M}{r^3} + \frac{1}{32M^2}\right]$$

$$= \left(768\pi M^2\right)^{-1}\left(1 - \frac{2M}{r}\right)^2\left[1 + \frac{4M}{r} + \frac{12M^2}{r^2}\right. \tag{27}$$

with T_{uv} and T_{uv} remaining as in (25).

Comparing (27) with (25) one finds that the effect of collapse is to add a constant term to T_{uu} which appears at large r as a flux of energy defined by Unruh [6] of magnitude $[768\pi M^2]^{-1*}$. This is just the energy flux one would expect on the basis of Hawking's arguments [1,2] as applied to this model.

From (27) and (25), one finds that the flux of energy is given by two components. Near the infinity it is dominated by an outward null flux of energy (given by T_{uu})*. Near the horizon, however, it is a flux of negative energy going into the horizon of the blackhole (represented by T_{vv} for r near $2M$)*

4. Hiscock's model of evaporating blackholes : calculation of stress tensor components

Hiscock [7] modelled the Hawking process of evaporation of a spherically symmetric blackhole with a Vaidya metric [8] which represents imploding null fluid. The metric of the model space-time is

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{28}$$

where

$$\begin{aligned} &M(v) = 0, \; v < 0, \\ &M(v) \neq 0, \; v_0 > v > 0, \\ &M(v) = 0, \; v > v_0. \end{aligned} \tag{29}$$

On the other hand, near $r = 2M$, $T_{vv} = -(768\pi M^2)^{-1}$

The model EBH (evaporating blackhole) space-time is initially flat, empty Minkowsi space for all $\upsilon < 0$. Then, at $\upsilon = 0$, a collapsing ball of mass $M(\upsilon) = m$ is formed. Negative-energy-density null fluid then falls into the hole at a greater or lesser rate, depending on the choice of $M(\upsilon)$, such that the mass of the blackhole is reduced to zero at $\upsilon = \upsilon_0$. The final state is again flat, empty Minkowski space for all $\upsilon > \upsilon_0$.

Hiscock took two examples for (29) in one of which

$$M(\upsilon) = \begin{cases} 0, & \upsilon < 0, \quad \text{Phase I} \\ m\left(1 - \dfrac{\upsilon}{\upsilon_0}\right), & \upsilon_0 > \upsilon > 0, \quad \text{Phase II} \\ 0, & \upsilon > \upsilon_0, \quad \text{Phase III} \end{cases} \tag{30}$$

so that with a θ = constant, ϕ = constant slice through the model FBH space-time to get a two-dimensional metric, we are left with

$$ds^2 = -\left[1 - \frac{2m}{r}\left(\frac{\upsilon_0 - \upsilon}{\upsilon_0}\right)\right]d\upsilon^2 + 2d\upsilon dr.. \tag{31}$$

With the following substitutions

$$z = \frac{\upsilon_0 - \upsilon}{r}, \quad \zeta = -\ln(\upsilon_0 - \upsilon)$$

$$\eta = \zeta + 2z^*, \quad z^* = \int_{\mu = m/\upsilon_0} (z^2 - 2\mu z^3 + 2z)^{-1} dz, \tag{32}$$

(31) reduces to the form

$$ds^2 = -e^{-2\zeta}\left(1 - 2\mu z + \frac{2}{z}\right)d\zeta d\eta. \tag{33}$$

This is the metric for phase II ($\upsilon_0 > \upsilon > 0$). The metric for phase I is

$$ds^2 = -du d\upsilon, \ (\upsilon < 0) \tag{34}$$

and, in phase III, the final Minkowski space-time is

$$ds^2 = -dU d\upsilon, \ (\upsilon > \upsilon_0). \tag{35}$$

The two-dimensional stress tensor for a quantised massless scalar may now be computed by relating these three sets of null co-ordinates eqs. (33–35) to the canonical set $(\bar{u}, \bar{\upsilon})$ in which the vacuum state is defined. The results are, *for phase I* ($\upsilon < 0$)

$$T_{\mu\nu} = 0,$$

for phase II ($\upsilon_0 > \upsilon > 0$):

$$T_{\eta\eta} = (12\pi)^{-1}\left(\frac{3}{4}\mu^2 z^4 - \frac{1}{2}\mu z^3 + \frac{6m\upsilon_0}{\bar{u}^2} - \frac{4m\upsilon_0^2}{\bar{u}^3} - \frac{12m^2\upsilon_0^2}{\bar{u}^4}\right. \tag{37}$$

$$T_{\zeta\zeta} = (12\pi)^{-1}\left(\frac{3}{4}\mu^2 z^4 - \frac{1}{2}\mu z^3\right), \tag{38}$$

$$T_{\zeta\eta} = -\frac{z^3}{24\pi}\left(1 - 2\mu z + \frac{2}{z}\right), \tag{39}$$

and *phase III* ($\upsilon > \upsilon_0$) :

$$T_{UU} = \frac{m\upsilon_0\left(3\bar{u}^2 - 2\upsilon_0\bar{u} - 6m\upsilon_0\right)}{6\pi\bar{u}^4\left(U-\upsilon_0\right)^2} \tag{40}$$

$$T_{\upsilon\upsilon} = T_{U\upsilon} = 0. \tag{41}$$

The stress tensor in phase II is observed to be finite everywhere except $z = +\infty$ and/or $\bar{u} = 0$ at the curvature singularity. The (η, ζ) co-ordinate system behaves poorly as $z \rightarrow z_+$, $\eta \rightarrow \infty$ (the event horizon), but examination of the stress tensor components in a Kruskal-type co-ordinate system regular on the event horizon shows that they are finite there.

The stress-energy in phase III consists solely of a stream of outgoing radiation whose energy density diverges as $U \rightarrow \upsilon_0$, *i.e.*, as one approaches the Cauchy horizon. ["The Cauchy horizon, simply, is the place where the Cauchy problem breaks down; usually it occurs accompanied by a naked singularity (a pathological causal structure)"— Kaminaga [9]]. Note also that the integrated energy density diverges as $U \rightarrow \upsilon_0$. The energy density is always positive for $U < \upsilon_0$. Since the stress tensor for phase II is finite all along the event horizon, it is natural to associate this diverging energy flux with the naked singularity.

Hiscock's model has been extended to an evaporating charged blackhole by Kaminaga [9].

5. Balbinot's formula for T_{VV}

Balbinot [10–13] extended the work of Davies *et al* and Hiscock to a physically general line-element describing a spherically symmetric evaporating blackhole of the form

$$ds^2 = -e^{2\psi}\left(1 - \frac{2m}{r}\right)d\upsilon^2 + 2e^{\psi}d\upsilon dr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{42}$$

where ψ and m are functions of υ and r. In four dimensions m is the total gravitational mass of the system as viewed from infinity. Taking a $\theta =$ const., $\phi =$ const. slice, (42) reduces to

$$ds^2 = -e^{2\psi}\left(1 - \frac{2m}{r}\right)d\upsilon^2 + 2e^{\psi}d\upsilon dr \tag{43}$$

If $\psi = 0$ and $m = M =$ const., (43) describes a two-dimensional Schwarzschild space-time in advanced time, Eddington-Finkelstein coordinates.

A new set of null coordinates (U, V) defined by

$$dU = g\left[dv - 2e^{-\psi}\left(1 - \frac{2m}{r}\right)^{-1} dr\right], \tag{44}$$

$$V = v$$

are introduced, where g is an integrating factor which satisfies

$$-\frac{1}{2}\frac{\partial g}{\partial r} = \frac{\partial}{\partial v}\left[ge^{-\psi}\left(1 - \frac{2m}{r}\right)^{-1}\right] \tag{45}$$

In terms of U, V, (43) becomes

$$ds^2 = -\frac{e^{2\psi}}{g}\left(1 - \frac{2m}{r}\right)dUdV. \tag{46}$$

which is manifestly conformally flat.

Now, in a two-dimensional space-time having a line-element

$$ds^2 = -C(\bar{u}, \bar{v})d\bar{u}d\bar{v} \tag{47}$$

the expectation value $T_{\mu\nu}$ of a massless scalar field in the vacuum state $|0\rangle$ defined by the normal modes, exp $(-i\omega\bar{u})$ and exp $(-i\omega\bar{v})$ is given by (6) and (7).

In general, the $|0\rangle$ does not represent the correct vacuum state for an evaporating blackhole so that one cannot simply use (6) as it stands. A prescription for how to define the correct vacuum state, call it $|\xi\rangle$, for an evaporating blackhole in a non-stationary space-time having a line-element (46) is not yet known but, for his purpose, Balbinot considered it sufficient to use some general properties of $\left(T_{\mu\nu}\right)_\xi$ for the spacetime (46).

In fact, requiring that ψ and m are well-behaved at past null infinity (*i.e.*, the space-time under consideration is past asymptatically flat) the scalar field modes for the $|\xi\rangle$ vacuum will have the form exp $(-i\omega V)$ on I^-. This gives the relation

$$\bar{v} = V \tag{48}$$

which is valid everywhere in the space-time. By (48), the ingoing normal modes for $|0\rangle$ and $|\xi\rangle$ vacua coincide, so $(T_{VV})_0 = (T_{VV})_\xi$ (in two dimensions there is no scattering of massless particles by the geometry). The outgoing modes do not contribute to T_{VV} and, by construction, both vacua reduce to the usual Minkowski vacuum on I^-. For the state $|\xi\rangle$ one must further require that the invariants (*e.g.*, $\langle T^{\mu\nu}\rangle\langle T_{\mu\nu}\rangle$) of $(T_{\mu\nu})_\xi$ be well-behaved on the event horizon of the blackhole. This condition requires $(T_{UV})_\xi$ and $(T_{UU})_\xi$ to vanish there; away from the horizon their form will depend on the exact definition of the fields outgoing normal modes for the state $|\xi\rangle$ and, of course, on g. Balbinot was only interested in finding the flux of negative energy going down the hole. Thus, it is sufficient to look for the VV component of $(T\mu v)_\xi$ and this does not depend either on the choice of the outgoing modes of the field or on g; it is fixed by the metric (43) and by the boundary condition (48).

From (46),

$$C^{-\frac{1}{2}} = \left[g\left(1 - \frac{2m}{r}\right)^{-1} e^{-2\psi}\right]^{\frac{1}{2}}. \tag{49}$$

Then
$$\left(C^{-\frac{1}{2}}\right)_{,VV} = \frac{1}{4}C^{-\frac{1}{2}}Q^2 - \frac{1}{2}C^{-\frac{1}{2}}\frac{\partial Q}{\partial V} \tag{50}$$

where
$$Q = \frac{\partial \psi}{\partial v} + \left(1 - \frac{2m}{r}\right)e^{\psi}\frac{\partial \psi}{\partial r} + e^{\psi}\left(\frac{m}{r^2} - \frac{1}{r}\frac{\partial m}{\partial r}\right) \tag{51}$$

remembering that

$$\frac{\partial}{\partial V} = \frac{\partial}{\partial v} + \frac{1}{2}e^{\psi}\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}. \tag{52}$$

Hence, the expression for $(T_{VV})_\xi$ given by (6) and (7) is

$$(T_{VV})_\xi = -(12\pi)^{-1}\left[\frac{1}{4}Q^2 - \frac{1}{2}\frac{\partial Q}{\partial V}\right] \tag{53}$$

and does not depend on g.

Following Bardeen [14], Balbinot chose ψ to be roughly constant and $m(v, r) = m(v)$ near $r = 2m$ so that, from (53), he got

$$(T_{VV})_\xi = (24\pi)^{-1}\left[\frac{\dot{m}}{r^2} - \frac{m}{r^3} + \frac{3}{2}\frac{m^2}{r^4}\right] \tag{54}$$

This reduces to the third equation of (25) obtained by Davies *et al* [3] for a collapsing shell if $\dot{m} = \frac{dm}{dv} = 0$. Furthermore, as previously stated one expects $(T_{UU})_\xi$ to vanish on the event horizon and to give a non-vanishing, positive, outgoing flux across time-like surface $r = 2m$ (the apparent horizon of the dynamical model, the event horizon being located somewhere inside it [15]).

One can associate with the flux (54) a blackbody temperature T which should be considered as the effective temperature of the hole, since by the energy conservation (which $T_{\mu\nu}$ satisfies) one expects this temperature to reflect the radiation content emitted at infinity. From (54) we have

$$T = \left(12|T_{VV}|\pi^{-1}\right)^{\frac{1}{4}}. \tag{55}$$

If, however, we have a metric in an arbitrary form

$$ds^2 = -A(v, r)dv^2 + 2B(v, r)dv\,dr \tag{56}$$

then we have T_{vv} in the form

$$T_{vv} = -(12\pi)^{-1}\left[\frac{P^2}{4} - \frac{1}{2}\left(\frac{\partial}{\partial v} + \frac{1}{2}\frac{A}{B}\frac{\partial}{\partial r}\right)P\right] + F(v) \tag{57}$$

where $$P = \frac{1}{B}\left[\frac{\partial B}{\partial v} + \frac{1}{2}\frac{\partial A}{\partial r}\right] \tag{58}$$

and $F(v)$ is a function of v to be determined by a boundary condition T_{vv} on past null infinity.

6. Some applications of Balbinot's formula

(a) Evaporating blackholes in the presence of inflation :

Mallet [16] has taken the following metric for a model for the dynamical evolution an evaporating blackhole in an inflationary universe :

$$ds^2 = -\left(1 - \frac{2M(v)}{r} - \chi^2 r^2\right)dv^2$$
$$+ 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{59}$$

where $M(v)$ is some decreasing mass function and χ the effective cosmological constant associated with the de Sitter inflationary phase of the universe.

The two-dimensional space-time associated with eq. (59) is obtained by taking $\theta =$ const. and $\phi =$ const. with the result

$$ds^2 = -\left[1 - \frac{2M(v)}{r} - \chi^2 r^2\right]dv^2 + 2dv\,dr. \tag{60}$$

Applying (56) – (58) to (60) leads at once to (with $F(v) = 0$)

$$T_{vv} = (24\pi)^{-1}\left\{\left[\frac{\dot{M}(v)}{r^2} - \frac{M(v)}{r^3} + \frac{3}{2}\frac{M^2(v)}{r^4}\right]\right.$$
$$\left. + \chi^2\left[\frac{3M(v)}{r} - \frac{1}{2}\right]\right\} \tag{61}$$

From (61), the following picture emerges. Near the event horizon of the blackhole, there is a negative-energy flux into the hole due to the first term in (61) and this is interpreted by an observer outside the event horizon as an evaporation of the hole. On the other hand, since

$\chi^2 > 0$, the second term indicates that the net effect of inflationary environment is the introduction of a positive energy-flux of radiation into the hole causing a slight decrease in the evaporation process.

(b) Evaporating blackholes with acceleration :

Recently, Krori *et al* [17] have studied the effect of acceleration on an evaporating blackhole by the stress tensor approach. The two-dimensional metric obtained by taking θ = const. and ϕ = const. is [18]

$$ds^2 = -Hdv^2 + 2dv\,dr, \tag{62}$$

where

$$H = 1 - \frac{2m(v)}{r} + 6Am(v)p + ArG_{,p}p - A^2r^2G(p), \tag{63}$$

A = acceleration parameter,

$$G(p) = 1 - p^2 - 2Am(v)p^3 = \sin^2\theta, \tag{64}$$

$$G_{,p} = -2p - 6Am(v)p^2. \tag{65}$$

Table 1. An estimate of T_{vv}.

θ	$r = r_{s\theta}$	(a) Contribution of the first term ( ) of (66)	(b) Contribution of the second term ( ) of (66)	Sum of (a) and (b)
0	$3.46\,m$	$-\frac{0.8353\dot{m}}{m^2} + \frac{6.84\times10^{-3}}{m^2}$	$+\frac{0.11\times10^{-3}}{m^2}$	$-\frac{0.8353\dot{m}}{m^2} + \frac{6.95\times10^{-3}}{m^2}$
$\pi/2$	$2.2\,m$	$-\frac{0.20661\dot{m}}{m^2} + \frac{14.9\times10^{-3}}{m^2}$	$-\frac{7.999\times10^{-3}}{m^2}$	$-\frac{0.20661\dot{m}}{m^2} + \frac{6.901\times10^{-3}}{m^2}$
π	$1.73\,m$	$-\frac{0.33412\dot{m}}{m^2} + \frac{12.8\times10^{-3}}{m^2}$	$-\frac{5.66\times10^{-3}}{m^2}$	$-\frac{0.33412\dot{m}}{m^2} + \frac{7.14\times10^{-3}}{m^2}$

Now, applying (56)–(58) to (62) leads at once to (with $F(v) = 0$)

$$T_{vv} = -(12\pi)^{-1}\left[\left(-\frac{\dot{m}}{2r^2} + \frac{m}{2r^3} - \frac{3}{4}\frac{m^2}{r^4}\right)\right.$$

$$+\left\{\frac{Am}{4r^2}G,p + A^2\left(\frac{(G,p)^2}{16} - \frac{m}{2r}G(p)\right)\right.$$

$$-\frac{A^3}{4}rG(p)G,p + \frac{A^4}{4}r^2G^2(p) + \frac{A^2}{4}G(p)\left(1 - \frac{2m}{T}\right)$$

$$\left.\left.+\frac{A}{4}(6mp + rG,p - Ar^2G(p)) \times \left(\frac{2m}{r^3} + A^2G(p)\right)\right\}\right] \qquad (66)$$

Near the Schwarzschild surface, *i.e.*, $r = r_{s\theta}$, the first term within circular brackets in (66) represents a negative energy flux into the hole. On the other hand, the second term within curly brackets is the contribution due to acceleration parameter A and varies with θ. As Table 1 will show, the net energy flux, T_{vv}, is negative. Hence, a net positive out-flow (*i.e.*, radiation) will occur in accordance with energy conservation.

For numerical estimates, we shall take $Am = \frac{1}{\sqrt{54}}$ and use some relevant data from Farhoosh and Zimmerman [18]. We shall consider three specific directions, $\theta = 0$, $\pi/2$ and π.

The table (Table 1) shows that for practical purposes, $T_{vv}(0)$, $T_{vv}(\pi/2)$ and $T_{vv}(\pi)$ (for $\theta = 0$, $\pi/2$ and π respectively) are equal for small $\dot{m}$ (Davies *et al* [3]). However, strictly speaking, $|T_{vv}(\pi)|$ appears to be maximum. The table also reveals an interesting feature. The contribution from the second term of (66) so tampers that from the first term that T_{vv} has practically the same value for $\theta = 0$, $\pi/2$ and π (for small $\dot{m}$.

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